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ORIGINAL PAPER

# Singular nonlinear three point BVPs arising in thermal explosion in a cylindrical reactor

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**Abstract** In this work we deal with a nonlinear three point singular boundary value problems (SBVPs), when the nonlinearity depends upon derivative. We establish the maximum principles for linear model. Prove some new inequalities based on Bessel and modified Bessel functions. Finally by using the Monotone Iterative Technique, we obtain some new existence results with well order and reverse order upper and lower solutions. The method developed in this paper can be used in computer algebra to compute solutions of real life problems.

**Keywords** Singular differential equation · Monotone iterative technique · Upper and lower solutions · Reverse order · Green's function · Bessel function

**Mathematics Subject Classification** 34B16 · 34B27 · 34B60

## 1 Introduction

The appropriate equation for the thermal balance between the heat generated by the chemical reaction and that conducted away can be written as

$$\nabla^2 u(P) = f(P, u(P), du(P)/dP), \tag{1}$$

after some simplification and due to geometric similarity, we arrive at the following differential equation

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$$- y''(x) - \frac{n}{x} y'(x) = f(x, y, x^n y'), \quad 0 < x < 1, \tag{2}$$

where $n$ corresponds to geometry of the vessel under consideration. In this work we consider the case when $n = 1$, i.e., the reaction is taking place in cylindrical vessel whose length is much greater than the radius. Thus we have the following singular differential equation

$$- y''(x) - \frac{1}{x} y'(x) = f(x, y, xy'), \quad 0 < x < 1. \tag{3}$$

Chamber [1] considered the case when $f(x, y, xy') = e^y$. His model was based on Arrhenius law.

In this case we consider cylindrical vessel and there is another concentric cylinder inside the cylindrical vessel which we can use to monitor the temperature inside the vessel. We consider the following three point boundary condition,

$$y'(0) = 0, \quad y(1) = \delta y(\eta), \tag{4}$$

where $\delta > 0$, $0 < \eta < 1$. The boundary condition at $x = 1$ is the temperature at the walls of outer cylinder which is related to the temperature at walls at $x = \eta$ of an interior cylinder by $y(1) = \delta y(\eta)$. This model can help us to maintain the required temperature interior to the vessel which is otherwise not possible.

Several other real life problems are governed by similar equations, e.g., Polytropic and Isothermal Gas Spheres [2], Electrohydrodynamics [3]). Lots of results are available for such applications and their generalizations for two point BVPs (see [4–12]).

Multipoint BVPs has been studied in detail for $n = 0$ (see [13–21]). To the best of our knowledge for $n \geq 1$, less results are found. Recently, Verma et al. [22] and Singh et al. [23] established the existence results for $n = 2$ and $n = 1$, respectively for $f \equiv f(x, y)$.

The prime objective of this work is to prove some new inequalities based on Bessel and modified Bessel functions and establish the new existence results for (3)–(4) in a region $D := \{(x, u, xv) \in [0, 1] \times R^2 : \beta_0(x) \leq u \leq \alpha_0(x)\}$ or $\widetilde{D} := \{(x, u, xv) \in [0, 1] \times R^2 : \alpha_0(x) \leq u \leq \beta_0(x)\}$ by using the monotone iterative method with upper and lower solutions that are reverse ordered and well ordered. The functions $\beta_0(x)$ and $\alpha_0(x)$ are called upper and lower solutions of nonlinear three point SBVPs, (3)–(4), respectively. The function $\beta_0(x)$ satisfies the differential inequalities $-(x\beta_0'(x))' \geq xf(x, \beta_0, x\beta_0')$, $\beta_0'(0) = 0$, $\beta_0(1) \geq \delta\beta_0(\eta)$, and the function $\alpha_0(x)$ satisfies the reverse differential inequalities. We further assume that

$(F_1)$ the function $f : D$ (or $\widetilde{D}$) $\to R$ is continuous on $D$ (or $\widetilde{D}$);

$(F_2)$ for all $(x, u_1, xv), (x, u_2, xv) \in D$ (or $\widetilde{D}$),

  $(a)$ when $\lambda > 0$, there exist a constant $M_1 \geq 0$ in the region $D$ such that

$$u_1 \leq u_2 \Longrightarrow f(x, u_2, xv) - f(x, u_1, xv) \leq M_1(u_2 - u_1);$$

  $(b)$ when $\lambda < 0$, there exist a constant $M_2 \geq 0$ in the region $\widetilde{D}$ such that

$$u_1 \leq u_2 \Longrightarrow f(x, u_2, xv) - f(x, u_1, xv) \geq -M_2(u_2 - u_1);$$

$(F_3)$ there exist $N \geq 0$ such that for all $(x, u, xv_1), (x, u, xv_2) \in D$ (or $\widetilde{D}$),

$$|f(x, u, xv_2) - f(x, u, xv_1)| \leq N|xv_2 - xv_1|.$$

We consider the following monotone iterative scheme for nonlinear three point SBVPs (3)–(4),

$$\begin{aligned} &-y''_{n+1}(x) - \frac{1}{x}y'_{n+1}(x) - \lambda y_{n+1}(x) \\ &\quad = f(x, y_n, xy'_n) - \lambda y_n(x), \quad y'_{n+1}(0) = 0, \quad y_{n+1}(1) = \delta y_{n+1}(\eta), \end{aligned} \tag{5}$$

where $\sup\left(\frac{\partial f}{\partial y}\right) = \lambda \in \mathbb{R} \setminus \{0\}$ and $f(x, y, xy')$ satisfies $(F_1)$, $(F_2)$ and $(F_3)$.

## 2 Preliminaries

### 2.1 The linear model

In this section we discuss the following linear model corresponding to the nonlinear three point SBVPs (3)–(4),

$$-(xy'(x))' - \lambda xy(x) = x\,h(x), \quad 0 < x < 1, \tag{6}$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta) + b, \tag{7}$$

where $h \in C(I)$, $I = [0, 1]$ and $b$ is any constant. Now by using the Lommel's transformation (see [23,24]), we obtain the following bounded solutions (near origin) of the corresponding homogeneous differential $xy'' + y' + \lambda xy(x)$ of (6), i.e.,

$$y_1(x, \lambda) = \begin{cases} J_0\left(x\sqrt{\lambda}\right), & \text{if } \lambda > 0; \\ I_0\left(x\sqrt{|\lambda|}\right), & \text{if } \lambda < 0, \end{cases} \tag{8}$$

defined in terms of Bessel's and Modified Bessel's functions.

### 2.2 Green's function

The solution of nonhomogeneous linear model (6)–(7) with the help of Green's functions are discussed in this subsection. We also define the constant sign of Green's functions and on the basis of sign of $\lambda$, we divide this subsection into the following two cases:

#### *2.2.1 Case-I : When $\lambda > 0$*

Assume that
$(H_0)$ : $0 < \lambda < y_{0,1}^2$, $\delta\, Y_0\left(\eta\sqrt{\lambda}\right) - Y_0\left(\sqrt{\lambda}\right) \leq 0$, $J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right) < 0.$
where $y_{0,1}$ is the first positive zero of $Y_0(x)$ (see Remark (2.1) of [23]).

**Lemma 2.1** (See Lemma 3.1 of [23]) *For* $0 < \lambda < y_{0,1}^2$ *we have the following inequality*

$$J_0\left(x\sqrt{\lambda}\right) Y_0\left(t\sqrt{\lambda}\right) - Y_0\left(x\sqrt{\lambda}\right) J_0\left(t\sqrt{\lambda}\right) \leq 0, \;\; 0 \leq t, \; x \leq 1,$$

*such that* $t \leq x$ *and* $x$ *is fixed.*

**Lemma 2.2** *Let* $y \in C^2(I)$ *be a solution of nonhomogeneous linear three point SBVPs* (6)–(7) *then*

$$y(x) = \frac{b\; J_0\left(x\sqrt{\lambda}\right)}{J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)} - \int_0^1 t\; G(x,t)h(t)dt. \tag{9}$$

*Here* $G(x,t)$ *is the solution of corresponding homogeneous linear differential equation, with homogeneous boundary conditions of linear three point SBVPs* (6)–(7)*, and defined as*

$$G(x,t) = \begin{cases} \dfrac{\pi J_0\left(x\sqrt{\lambda}\right)\left(J_0\left(t\sqrt{\lambda}\right)\left(\delta Y_0\left(\eta\sqrt{\lambda}\right)-Y_0\left(\sqrt{\lambda}\right)\right)+Y_0\left(t\sqrt{\lambda}\right)\left(J_0\left(\sqrt{\lambda}\right)-\delta J_0\left(\eta\sqrt{\lambda}\right)\right)\right)}{2\left(J_0\left(\sqrt{\lambda}\right)-\delta J_0\left(\eta\sqrt{\lambda}\right)\right)}, & 0 \leq x \leq t \leq \eta; \\ \dfrac{\pi J_0\left(t\sqrt{\lambda}\right)\left(J_0\left(x\sqrt{\lambda}\right)\left(\delta Y_0\left(\eta\sqrt{\lambda}\right)-Y_0\left(\sqrt{\lambda}\right)\right)+Y_0\left(x\sqrt{\lambda}\right)\left(J_0\left(\sqrt{\lambda}\right)-\delta J_0\left(\eta\sqrt{\lambda}\right)\right)\right)}{2\left(J_0\left(\sqrt{\lambda}\right)-\delta J_0\left(\eta\sqrt{\lambda}\right)\right)}, & t \leq x, \; t \leq \eta; \\ \dfrac{\pi J_0\left(x\sqrt{\lambda}\right)\left(J_0\left(\sqrt{\lambda}\right)Y_0\left(t\sqrt{\lambda}\right)-Y_0\left(\sqrt{\lambda}\right)J_0\left(t\sqrt{\lambda}\right)\right)}{2\left(J_0\left(\sqrt{\lambda}\right)-\delta J_0\left(\eta\sqrt{\lambda}\right)\right)}, & x \leq t, \; \eta \leq t; \\ \dfrac{\pi\left(J_0\left(x\sqrt{\lambda}\right)\left(\delta J_0\left(\eta\sqrt{\lambda}\right)Y_0\left(t\sqrt{\lambda}\right)-Y_0\left(\sqrt{\lambda}\right)J_0\left(t\sqrt{\lambda}\right)\right)+\left(J_0\left(\sqrt{\lambda}\right)-\delta J_0\left(\eta\sqrt{\lambda}\right)\right)\left(J_0\left(t\sqrt{\lambda}\right)Y_0\left(x\sqrt{\lambda}\right)\right)\right)}{2\left(J_0\left(\sqrt{\lambda}\right)-\delta J_0\left(\eta\sqrt{\lambda}\right)\right)}, & \eta \leq t \leq x \leq 1, \end{cases}$$

*and if* $(H_0)$ *holds then* $G(x,t) \geq 0$.

*Proof* See the proof of Lemmas 3.2 and 3.3 of [23].

*2.2.2 Case-II : when* $\lambda < 0$

Assume that

$$(H_0') : \; \lambda < 0, \; \delta\; K_0\left(\eta\sqrt{|\lambda|}\right) - \; K_0\left(\sqrt{|\lambda|}\right) \geq 0, \;\; I_0\left(\sqrt{|\lambda|}\right) - \delta\; I_0\left(\eta\sqrt{|\lambda|}\right) > 0.$$

**Lemma 2.3** (See Lemma 3.4 of [23]) *For sufficiently small* $\lambda < 0$ *we have the following inequality*

$$I_0\left(t\sqrt{|\lambda|}\right) K_0\left(x\sqrt{|\lambda|}\right) - I_0\left(x\sqrt{|\lambda|}\right) K_0\left(t\sqrt{|\lambda|}\right) \leq 0, \;\;\; 0 \leq t, \; x \leq 1,$$

*such that* $t \leq x$ *and* $x$ *is fixed.*

**Lemma 2.4** *Let* $\lambda < 0$ *and* $y \in C^2(I)$ *be a solution of nonhomogeneous linear three point SBVPs* (6)–(7) *then*

$$y(t) = \frac{b\, I_0\left(x\sqrt{|\lambda|}\right)}{I_0\left(\sqrt{|\lambda|}\right) - \delta I_0\left(\eta\sqrt{|\lambda|}\right)} - \int_0^1 t\, G(x,t)h(t)dt. \tag{10}$$

*Here $G(x,t)$ is the solution of corresponding homogeneous linear differential equation, with homogeneous boundary conditions of linear three point SBVPs (6)–(7), and defined as*

$$G(x,t) = \begin{cases} \frac{I_0(x\sqrt{|\lambda|})(K_0(t\sqrt{|\lambda|})(\delta I_0(\eta\sqrt{|\lambda|})-I_0(\sqrt{|\lambda|}))+I_0(t\sqrt{|\lambda|})(K_0(\sqrt{|\lambda|})-\delta K_0(\eta\sqrt{|\lambda|})))}{I_0(\sqrt{|\lambda|})-\delta I_0(\eta\sqrt{|\lambda|})}, & 0 \le x \le t \le \eta; \\ \frac{I_0(t\sqrt{|\lambda|})(I_0(x\sqrt{|\lambda|})(K_0(\sqrt{|\lambda|})-\delta K_0(\eta\sqrt{|\lambda|}))-K_0(x\sqrt{|\lambda|})(I_0(\sqrt{|\lambda|})-\delta I_0(\eta\sqrt{|\lambda|})))}{I_0(\sqrt{|\lambda|})-\delta I_0(\eta\sqrt{|\lambda|})}, & t \le x, t \le \eta; \\ \frac{I_0(x\sqrt{|\lambda|})(K_0(\sqrt{|\lambda|})I_0(t\sqrt{|\lambda|})-I_0(\sqrt{|\lambda|})K_0(t\sqrt{|\lambda|}))}{I_0(\sqrt{|\lambda|})-\delta I_0(\eta\sqrt{|\lambda|})}, & x \le t, \eta \le t; \\ \frac{I_0(x\sqrt{|\lambda|})(K_0(\sqrt{|\lambda|})I_0(t\sqrt{|\lambda|})-\delta I_0(\eta\sqrt{|\lambda|})K_0(t\sqrt{|\lambda|}))-(I_0(\sqrt{|\lambda|})-\delta I_0(\eta\sqrt{|\lambda|}))(I_0(t\sqrt{|\lambda|})K_0(x\sqrt{|\lambda|}))}{I_0(\sqrt{|\lambda|})-\delta I_0(\eta\sqrt{|\lambda|})}, & \eta \le t \le x \le 1, \end{cases}$$

*and if $\left(H_0'\right)$ holds then $G(x,t) \le 0$.*

## 3 Maximum and anti maximum principles

**Proposition 3.1** *Suppose $(H_0)$ holds, such that $y \in C^2(I)$ and $y$ satisfies*

$$\begin{aligned} &-(xy'(x))' - \lambda xy(x) \ge 0, \quad 0 < x < 1, \\ &y'(0) = 0, \quad y(1) \ge \delta y(\eta), \end{aligned}$$

*then $y(x) \le 0$, $\forall x \in [0,1]$.*

*Proof* By using the equation (9) with assumption $(H_0)$ and constant sign of Green function ($G \ge 0$), we can easily prove that $y(x) \le 0$.

Similarly, by using the equations (10) and assumption $(H_0')$, we can prove the following Proposition.

**Proposition 3.2** *Suppose $(H_0')$ holds, $y \in C^2(I)$ and $y$ satisfies*

$$\begin{aligned} &-(xy'(x))' - \lambda xy(x) \ge 0, \quad 0 < x < 1, \\ &y'(0) = 0, \quad y(1) \ge \delta y(\eta), \end{aligned}$$

*then $y(x) \ge 0$, $\forall x \in [0,1]$.*

## 4 Inequalities and existence results

In this section we discuss our main results. We prove some new inequalities based upon Bessel and Modified Bessel function and establish the new existence results for both cases, i.e., when upper and lower solutions are well ordered or in reverse order. We divide this section into the following two subsections.

4.1 Reverse ordered lower and upper solutions ($\alpha_0 \geq \beta_0$)

**Lemma 4.1** *If* $0 < \lambda < y_{0,1}^2$*, then the Bessel functions* $J_0$ *and* $J_1$ *satisfy the following inequality*

$$(\lambda - M_1)J_0(x\sqrt{\lambda}) - Nx\sqrt{\lambda}J_1(x\sqrt{\lambda}) \geq 0,$$

*for all* $x \in [0, 1]$*, whenever*

$$\lambda \geq M_1 + \frac{N^2}{2} + \frac{N}{2}\sqrt{N^2 + 4M_1}\,, \tag{11}$$

*such that* $M_1, N \in R^+$.

*Proof* When $0 < \lambda < y_{0,1}^2$, the Bessel functions satisfy the inequality $J_0(x\sqrt{\lambda}) \geq J_1(x\sqrt{\lambda})$, for all $x \in [0, 1]$, which gives us

$$(\lambda - M_1)J_0(x\sqrt{\lambda}) - Nx\sqrt{\lambda}J_1(x\sqrt{\lambda}) \geq \left((\lambda - M_1) - N\sqrt{\lambda}\right) J_0(x\sqrt{\lambda}).$$

Now right hand side will be positive provided $\left((\lambda - M_1) - N\sqrt{\lambda}\right) \geq 0$, which gives $\lambda \geq M_1 + \frac{N^2}{2} + \frac{N}{2}\sqrt{N^2 + 4M_1}$. Hence the lemma.

*Remark 4.1* It is clear that $G(x, t) \geq 0$, for all $x, t \in [0, 1]$, when $(H_0)$ holds. As $G(x, t)$ satisfies

$$-(xG'(x))' - \lambda x G(x) = 0, \quad 0 < x < 1,$$
$$G'(0) = 0, \quad G(1) = \delta G(\eta),$$

we deduce that $G'(x, t) \leq 0$ and $xG'(x, t) \geq \frac{\lambda}{\lambda - 1}G(x, t)$ for $\lambda < 1$.

**Lemma 4.2** *Suppose* $(H_0)$ *holds and such that* $1 > \lambda \geq M_1$ *then for all* $x, t \in [0, 1]$*, we have the inequality*

$$(\lambda - M_1)G(x, t) + N\; x\; (sign\; y')\frac{\partial G(x, t)}{\partial x} \geq 0,$$

*whenever* $(\lambda - M_1) - N\frac{\lambda}{1-\lambda} \geq 0$ *and* $M_1, N \in R^+$.

*Proof* From the above Remark 4.1, it is clear that to prove the above inequality, it is sufficient to prove $(\lambda - M_1)G(x, t) + N\, x\, \frac{\partial G(x,t)}{\partial x} \geq 0$. Now again by using Remark 4.1, we can write

$$(\lambda - M_1)G(x, t) + N\; x\; \frac{\partial G(x, t)}{\partial x} \geq \left((\lambda - M_1) - N\frac{\lambda}{1 - \lambda}\right) G(x, t). \tag{12}$$

Now if $(\lambda - M_1) - N\frac{\lambda}{1-\lambda} \geq 0$, then right hand side will we positive. This completes the lemma.

*Remark 4.2* If $y_n = \alpha_{n+1} - \alpha_n$, and $f$ is defined in domain $D$, then we observe that

$$-(xy_n')' - \lambda x y_n = xf(x, \alpha_n, x\alpha_n') + (x\alpha_n')', \tag{13}$$
$$y_n'(0) = 0, \quad y_n(1) = \delta y_n(\eta) \tag{14}$$

and if we assume that $\alpha_n$ is lower solution of (3)–(4), then (13)–(14) are reduced into the following SBVP

$$-(xy_n')' - \lambda x y_n = xf(x, \alpha_n, x\alpha_n') + (x\alpha_n')' \geq 0,$$
$$y_n'(0) = 0, \quad y_n(1) \geq \delta y_n(\eta).$$

Finally, by using the Proposition 3.1, we get $y_n \leq 0$, i.e., $\alpha_{n+1} \leq \alpha_n$. Similarly we can get $\beta_{n+1} \geq \beta_n$, where $\beta_n$ is an upper solution of (3)–(4).

**Proposition 4.1** *Suppose* $(H_0)$ *holds, the source function* $f$ *satisfies* $(F_1)$, $(F_2)$ *and* $(F_3)$ *and there exist* $0 < \max\{M_1, M_1 + \frac{N^2}{2} + \frac{N}{2}\sqrt{N^2 + 4M_1}\} \leq \lambda < 1$, *such that* $(\lambda - M_1) - N\frac{\lambda}{1-\lambda} \geq 0$ *is valid. Then the functions* $\alpha_n$ *and* $\beta_n$, *satisfy the following relations*

$(a)$ $\alpha_{n+1} \leq \alpha_n$, *for all* $n \in \mathbb{N}$, *where* $\alpha_n$ *is a lower solution of* (3)–(4),
$(b)$ $\beta_{n+1} \geq \beta_n$, *for all* $n \in \mathbb{N}$, *where* $\beta_n$ *is an upper solution of* (3)–(4),

*and* $\alpha_n$, $\beta_n$ *are defined recursively by (5).*

*Proof* The above claim is proved by using the principle of Mathematical Induction. Claim $(a)$ holds for $n = 0$, i.e., $\alpha_1 \leq \alpha_0$ (see Remark (4.2)). Now suppose that claim is true for $n-1$, i.e., $\alpha_n \leq \alpha_{n-1}$, where $\alpha_{n-1}$ is lower solution of (3)–(4), and we will show that the claim is true for $n$.

Let $y = \alpha_n - \alpha_{n-1}$, then it is clear that $y$ satisfies

$$-(xy')' - \lambda x y = (x\alpha_{n-1}')' + xf(x, \alpha_{n-1}, x\alpha_{n-1}') \geq 0, \tag{15}$$
$$y'(0) = 0, \quad y(1) \geq \delta y(\eta). \tag{16}$$

To show that $\alpha_{n+1} \leq \alpha_n$, we have to prove that $\alpha_n$ is a lower solution of (3)–(4), i.e.,

$$-(x\alpha_n')' - xf(x, \alpha_n, x\alpha_n') \leq x\left[(\lambda - M_1)y + N(sign\ y')xy'\right], \tag{17}$$

where right hand side should be negative. Now, by using equation (9) it is sufficient to prove

$$(\lambda - M_1)J_0(x\sqrt{\lambda}) - Nx\sqrt{\lambda}J_1(x\sqrt{\lambda}) \geq 0,$$
$$(\lambda - M_1)G(x, t) + N\ x\ (sign\ y')\frac{\partial G(x, t)}{\partial x} \geq 0,$$

for all $x, t \in [0, 1]$. Which are true by Lemmas 4.1 and 4.2. Hence $\alpha_{n+1} \leq \alpha_n$.

Using similar analysis we can prove the claim $(b)$. Hence $\beta_{n+1} \geq \beta_n$.

**Proposition 4.2** *Suppose* $(H_0)$ *holds, the source term* $f$ *satisfies* $(F_1)$, $(F_2)$ *and* $(F_3)$ *and there exist* $0 < \max\{M_1, M_1 + \frac{N^2}{2} + \frac{N}{2}\sqrt{N^2+4M_1}\} \leq \lambda < 1$ *such that* $(\lambda - M_1) - N\frac{\lambda}{1-\lambda} \geq 0$ *and for all* $x \in [0, 1]$

$$f(x, \beta(x), x\beta'(x)) - f(x, \alpha(x), x\alpha'(x)) - \lambda(\beta - \alpha) \geq 0,$$

*is valid. Then for all* $n \in \mathbb{N}$, *the functions* $\alpha_n$ *and* $\beta_n$ *defined by (5), satisfy* $\alpha_n \geq \beta_n$.

*Proof* Suppose $y_i = \beta_i - \alpha_i$, it is clear that $y_i$ satisfies the singular differential equation

$$\begin{aligned} -(xy_i')' - x\lambda y_i &= x\left[f(x, \beta_{i-1}(x), x\beta_{i-1}'(x)) - f(x, \alpha_{i-1}(x), x\alpha_{i-1}'(x))\right.\\ &\quad \left.-\lambda(\beta_{i-1} - \alpha_{i-1})\right], \quad (18)\\ &= x[h_{i-1}], \quad (19)\end{aligned}$$

where $h_{i-1} = f(x, \beta_{i-1}(x), x\beta_{i-1}'(x)) - f(x, \alpha_{i-1}(x), x\alpha_{i-1}'(x)) - \lambda(\beta_{i-1} - \alpha_{i-1})$.

To prove $\beta_i \leq \alpha_i$, for all $i \in \mathbb{N}$, we have to show that $h_{i-1} \geq 0$, for all $i \in \mathbb{N}$. We use Mathematical Induction. For $i = 1$, the equation (18) is reduced into

$$\begin{aligned} -(xy_1')' - x\lambda y_1 &= x\left[f(x, \beta_0(x), x\beta_0'(x)) - f(x, \alpha_0(x), x\alpha_0'(x)) - \lambda(\beta_0 - \alpha_0)\right],\\ &= x[h_0],\end{aligned}$$

by using the conditions $(F_2)$ and $(F_3)$, we can easily show that $h_0 \geq 0$, and $y_1'(0) = 0$, $y_1(1) = \delta y_1(\eta)$. Using Proposition 3.1, we deduce that $y_1 \leq 0$, i.e., $\beta_1 \leq \alpha_1$. Now suppose $h_{n-2} \geq 0$ and $\beta_{n-1} \leq \alpha_{n-1}$, and we have to prove that $h_{n-1} \geq 0$ and $\beta_n \leq \alpha_n$.

As

$$\begin{aligned} h_{n-1} &= f(x, \beta_{n-1}(x), x\beta_{n-1}'(x)) - f(x, \alpha_{n-1}(x), x\alpha_{n-1}'(x)) - \lambda(\beta_{n-1} - \alpha_{n-1}), \quad (20)\\ &= -\left[(\lambda - M_1)y_{n-1} + N(sign\ y_{n-1}')xy_{n-1}'\right]. \quad (21)\end{aligned}$$

where $y_{n-1} = \beta_{n-1} - \alpha_{n-1}$ is the solutions of nonhomogeneous linear BVP (6)–(7), with $h_{n-2} \geq 0$ and $y_{n-1}'(0) = 0$, $y_{n-1}(1) = \delta y_{n-1}(\eta)$. We can follow the same analysis as we did in Proposition 4.1 and we have $h_{n-1} \geq 0$, and $y_n'(0) = 0$, $y_n(1) = \delta y_n(\eta)$. By using Proposition 3.1, we deduce that $y_n \leq 0$, i.e., $\alpha_n \geq \beta_n$.

#### *4.1.1 Priori's bound*

**Lemma 4.3** *If* $f(x, y, xy')$ *satisfies*

$(F_R)$ *For all* $(x, y, xy') \in D$, $|f(x, y, xy')| \leq \varphi(|xy'|)$; *where* $\varphi : R^+ \to R^+$ *is continuous and satisfies*

$$\frac{1}{2} < \int_{l_0}^{\infty} \frac{ds}{\varphi(s)},$$

*where* $l_0 = \sup\limits_{[0,1]} 2\,|x\alpha_0(x)|$, *then there exists* $R > 0$ *such that any solution of*

$$-(xy'(x))' \geq xf(x, y, xy'), \quad 0 < x < 1, \tag{22}$$

$$y'(0) = 0, \quad y(1) \geq \delta y(\eta), \tag{23}$$

*with* $y \in [\beta_0(x), \alpha_0(x)]$ *satisfies* $\|xy'\|_\infty \leq R$.

*Proof* We can divide this proof into following three cases:

Case : $(i)$ Suppose that the nature of the solution of nonlinear three point SBVP (3)–(4) is non monotone throughout the interval. First we consider the interval $(x_0, x] \in [0, 1]$, and assume that the slope of the solution at point $x_0$ is zero, and $y'(x) > 0$, for all $x > x_0$. Integrating the equation (22) from $x_0$ to $x$, we get

$$\int_0^{xy'} \frac{ds}{\varphi(s)} \leq \frac{1}{2}.$$

We choose $R$ such that

$$\int_0^{xy'} \frac{ds}{\varphi(s)} \leq \frac{1}{2} < \int_{l_0}^{R} \frac{ds}{\varphi(s)} \leq \int_0^{R} \frac{ds}{\varphi(s)}.$$

This gives $xy'(x) \leq R$.

Now suppose that the slope of the solution at point $x_0$ is zero, and $y'(x) < 0$, for all $x < x_0$. Following the same analysis (as we did above), we get $-xy'(x) \leq R$.

Case : $(ii)$ Suppose the nature of the solution of nonlinear three point SBVP (3)–(4) is monotonically increasing throughout the interval, i.e., $y'(x) > 0$ in $(0, 1)$, then (by using Mean value Theorem) $\exists$ a $\tau \in (0, 1)$, such that

$$y'(\tau) = \frac{y(1) - y(0)}{1 - 0} \leq 2|\alpha_0|.$$

Integrating the equation (22) from $\tau$ to $x$ and then using the assumption $(F_R)$ we get,

$$\int_0^{xy'} \frac{ds}{\varphi(s)} \leq \frac{1}{2} + \int_0^{l_0} \frac{ds}{\varphi(s)} < \int_0^{R} \frac{ds}{\varphi(s)},$$

which gives $xy'(x) \leq R$.

Similarly, when $y$, i.e., the solution of nonlinear three point SBVP (3)–(4) is monotonically decreasing throughout the interval, then we get $-xy'(x) \leq R$.

Similarly we can prove the following result.

**Lemma 4.4** *If* $f(x, y, xy')$ *satisfies* $(F_R)$, *then there exists* $R > 0$ *such that any solution of*

$$-(xy'(x))' \leq xf(x, y, xy'), \quad 0 < x < 1, \tag{24}$$

$$y'(0) = 0, \quad y(1) \leq \delta y(\eta), \tag{25}$$

*with* $y \in [\beta_0(x), \alpha_0(x)]$ *satisfies* $\|xy'\|_\infty \leq R$.

**Theorem 4.1** *Suppose* $(H_0)$ *holds, the source term* $f$ *satisfies* $(F_1)$, $(F_2)$ *and* $(F_3)$ *and there exist* $\lambda > 0$ *such that* $1 > \lambda \geq \max\{M_1, M_1 + \frac{N^2}{2} + \frac{N}{2}\sqrt{N^2 + 4M_1}\}$ *and* $(\lambda - M_1) - N\frac{\lambda}{1-\lambda} \geq 0$, *and for all* $x \in [0, 1]$

$$f(x, \beta(x), x\beta'(x)) - f(x, \alpha(x), x\alpha'(x)) - \lambda(\beta - \alpha) \geq 0,$$

*is valid, then the sequences* $(\alpha_n)$ *and* $(\beta_n)$ *defined by* (5), *starting with* $\alpha$ *and* $\beta$ *as initial guesses, converge monotonically in* $C^1([0, 1])$ *to solution* $v$ *and* $u$ *of nonlinear BVP* (3)–(4), *such that for all* $x \in [0, 1]$, $\beta \leq u \leq v \leq \alpha$. *Any solution* $z(x)$ *of* (3)–(4) *in* $D$ *satisfies* $u(x) \leq z(x) \leq v(x)$.

*Proof* We can easily show that

$$\alpha = \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n \geq \cdots \geq \beta_n \geq \cdots \geq \beta_1 \geq \beta_0 = \beta. \tag{26}$$

with the help of Propositions 4.1 and 4.2, it is clear that $\{\alpha_n\}$ and $\{\beta_n\}$ are monotone and bounded. Now by using Dini's theorem these sequences converges uniformly. Suppose $\alpha_n \to v$ and $\beta_n \to u$. By using Priori bound and $(F_1)$, we can find that the sequences $\{x\alpha'_n\}$ and $\{x\beta'_n\}$ are equibounded and equicontinuous in $C^1([0, 1])$, i.e., there exist uniformly convergent subsequences $\{x\alpha'_{nk}\}$ and $\{x\beta'_{nk}\}$ in $C^1([0, 1])$ (Arzela-Ascoli Theorem). It is easy to check that $x\alpha'_n \longrightarrow xv'$ and $x\beta'_n \longrightarrow xu'$, whenever $\alpha_n \to v$ and $\beta_n \longrightarrow u$.

As equation (9) represents the solution of (5) with $h(x) = f(x, y_n, xy_n) - \lambda y_n$. By taking limit as $n \to \infty$ on both sides of (9), we get that $v$ and $u$ are solutions of nonlinear three point SBVPs (3)–(4). Any solution $z(x)$ in $D$ plays the role of $\alpha_0$, i.e., $z(x) \leq v(x)$. Similarly we get $z(x) \geq u(x)$.

### 4.2 Well-ordered lower and upper solutions ($\alpha_0 \leq \beta_0$)

**Lemma 4.5** *Let* $\lambda < 0$, *then Modified Bessel functions* $I_0$ *and* $I_1$ *satisfy the following inequality*

$$(\lambda + M_2)I_0(x\sqrt{|\lambda|}) + Nx\sqrt{|\lambda|}I_1(x\sqrt{|\lambda|}) \leq 0,$$

*for all* $x \in [0, 1]$ *if* $\lambda$ *satisfies*

$$\lambda \leq -M_2 - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M_2}, \tag{27}$$

*where* $M_2, N \in R^+$.

*Proof* When $\lambda < 0$, the Modified Bessel's function $I_0$ and $I_1$ satisfy the inequality $I_0(x\sqrt{|\lambda|}) \geq I_1(x\sqrt{|\lambda|})$, for all $x \in [0, 1]$, which gives

$$(\lambda + M_2)I_0(x\sqrt{|\lambda|}) + Nx\sqrt{|\lambda|}I_1(x\sqrt{|\lambda|}) \leq \left((M_2 + \lambda) + N\sqrt{|\lambda|}\right) I_0(x\sqrt{|\lambda|}).$$

It is clear that we get the required solution provided $(M_2 + \lambda) + N\sqrt{|\lambda|} \leq 0$, i.e.,

$$\lambda \leq -M_2 - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M_2}.$$

*Remark 4.3* By argument similar to Remark 4.1, we get $G'(x, t) \leq 0$ and $-xG'(x, t) \leq \lambda G(x, t)$.

**Lemma 4.6** *Suppose* $(H_0')$ *holds and* $\lambda < 0$ *such that* $\lambda + M_2 \leq 0$*, then for all* $x, t \in [0, 1]$*, we have the inequality*

$$(\lambda + M_2)G(x, t) + N\, x\, (sign\ y')\frac{\partial G(x, t)}{\partial x} \geq 0,$$

*whenever* $(\lambda + M_2) - N\lambda \leq 0$ *such that* $M_2, N \in R^+$.

*Proof* See the proof of Lemma 4.2, with Remark 4.3.

*Remark 4.4* By arguments, similar to Remark 4.2, we can show that $\alpha_{n+1} \geq \alpha_n$ and $\beta_{n+1} \leq \beta_n$, in $\widetilde{D}$.

**Proposition 4.3** *Suppose* $(H_0')$ *holds,* $f$ *satisfies* $(F_1)$, $(F_2)$ *and* $(F_3)$ *and there exist* $\lambda < 0$ *such that* $\lambda \leq \min\{-M_2, -M_2 - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M_2}, -\frac{M_2}{1-N}\}$*, then the functions* $\alpha_n$ *and* $\beta_n$*, satisfy the following relations*

($a$) $\alpha_{n+1} \geq \alpha_n$*, for all* $n \in \mathbb{N}$*, where* $\alpha_n$ *is lower solution of (3)–(4),*
($b$) $\beta_{n+1} \leq \beta_n$*, for all* $n \in \mathbb{N}$*, where* $\beta_n$ *is an upper solution of (3)–(4),*

*where* $\alpha_n$ *and* $\beta_n$ *are defined recursively by (5).*

*Proof* See the proof of Proposition 4.1 with Lemmas 4.5, 4.6 and Remark 4.4.

**Proposition 4.4** *Suppose* $(H_0')$ *holds, the source term* $f$ *satisfies* $(F_1)$, $(F_2)$ *and* $(F_3)$ *and* $\lambda < 0$ *such that* $\lambda \leq \min\{-M_2, -M_2 - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M_2}, -\frac{M_2}{1-N}\}$*, and for all* $x \in [0, 1]$

$$f(x, \beta(x), x\beta'(x)) - f(x, \alpha(x), x\alpha'(x)) - \lambda(\beta - \alpha) \geq 0,$$

*is valid. Then for all* $n \in \mathbb{N}$*, the functions* $\alpha_n$ *and* $\beta_n$ *defined by (5), satisfy* $\alpha_n \leq \beta_n$.

*Proof* Proof is similar to the proof of Proposition 4.2.

**Lemma 4.7** *If $f(x, y, xy')$ satisfies*

($F_W$) *For all $(x, y, xy') \in \widetilde{D}$, $|f(x, y, xy')| \leq \varphi(|xy'|)$; where $\varphi : R^+ \to R^+$ is continuous and satisfies*

$$\frac{1}{2} < \int_{l_0}^{\infty} \frac{ds}{\varphi(s)},$$

*where $l_0 = \sup_{[0,1]} 2\,|x\beta_0(x)|$, then there exists $R > 0$ such that any solution of*

$$-(xy'(x))' \geq xf(x, y, xy'), \quad 0 < x < 1, \tag{28}$$

$$y'(0) = 0, \quad y(1) \geq \delta y(\eta), \tag{29}$$

*with $y \in [\alpha_0(x), \beta_0(x)]$ satisfies $\|xy'\|_\infty \leq R$.*

**Lemma 4.8** *If $f(x, y, xy')$ satisfies ($F_W$), then there exists $R > 0$ such that any solution of*

$$-(xy'(x))' \leq xf(x, y, xy'), \quad 0 < x < 1, \tag{30}$$

$$y'(0) = 0, \quad y(1) \leq \delta y(\eta), \tag{31}$$

*with $y \in [\alpha_0(x), \beta_0(x)]$ satisfies $\|xy'\|_\infty \leq R$.*

**Theorem 4.2** *Suppose ($H_0'$) holds, the source term $f(x, y, xy')$ satisfies ($F_1$), ($F_2$) and ($F_3$) and $\lambda < 0$ be such that $\lambda \leq \min\{-M_2, -M_2 - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M_2}, -\frac{M_2}{1-N}\}$, and for all $x \in [0, 1]$*

$$f(x, \beta(x), x\beta'(x)) - f(x, \alpha(x), x\alpha'(x)) - \lambda(\beta - \alpha) \geq 0,$$

*is valid, then the sequences $(\alpha_n)$ and $(\beta_n)$ defined by (5), , starting with $\alpha$ and $\beta$ as initial guesses, converge monotonically in $C^1([0, 1])$ to solution $v$ and $u$ of nonlinear BVP (3)–(4), such that for all $x \in [0, 1]$, $\alpha \leq v \leq u \leq \beta$. Any solutions $z(x)$ of (3)–(4) in $\widetilde{D}$ satisfy $v(x) \leq z(x) \leq u(x)$.*

*Proof* The proof of this Theorem follows same analysis as we did in Theorem 4.1.

## 5 Examples

Numerical Examples are discussed in this section which help us to validate our analytical results, and show that $\exists$ a $\lambda \in \mathbb{R} \setminus \{0\}$ which satisfies the sufficient conditions of Theorems 4.1, and 4.2.

*Example 5.1* Consider the nonlinear three point SBVP

$$-y''(x) - \frac{1}{x}y'(x) = \frac{(y(x))^3}{80} + \frac{xy'}{7} + \frac{\sin x}{160}, \tag{32}$$

$$y'(0) = 0, \quad y(1) = 3y\left(\frac{1}{4}\right). \tag{33}$$

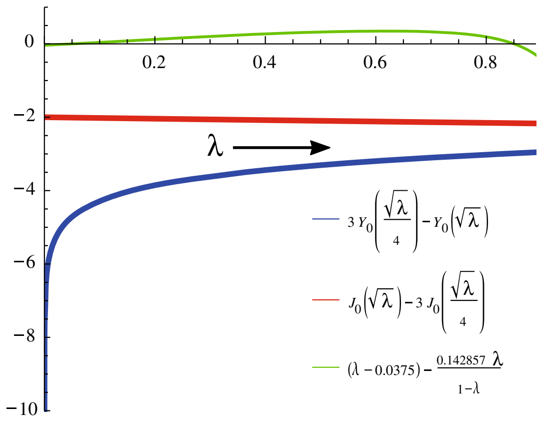



**Fig. 1** Plot of $(H_0)$ and $(\lambda - M_1) - N\frac{\lambda}{1-\lambda}$

Here solution of nonlinear three point SBVP (32)–(33) has $\alpha_0 = 1$ and $\beta_0 = -1$ as lower and upper bounds, respectively. This is a reverse ordered case. The nonlinear sources term satisfies the conditions $(F_1)$, $(F_2)$ and $(F_3)$ in domain $D$. Here Lipschitz constant are $M_1 = \frac{3}{80}$ and $N = \frac{1}{7}$. From Fig. 1 it is clear that we can find out a range of $\lambda > 0$ such that

$$\left(\max\left\{M_1, \sup\left(\frac{f(x,\beta,x\beta') - f(x,\alpha,x\alpha')}{\beta-\alpha}\right), M_1 + \frac{N^2}{2} + \frac{N}{2}\sqrt{N^2+4M_1}\right\} < \lambda < y_{0,1}^2\right),$$

i.e., $0.0771902 \leq \lambda < 1$. So that $(H_0)$, and $(\lambda - M_1) - N\frac{\lambda}{1-\lambda} \geq 0$ are satisfied. Thus Theorem 4.1 is applicable here.

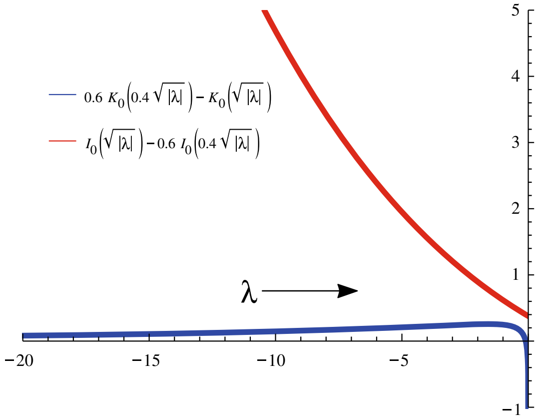
*Example 5.2* Consider the nonlinear three point SBVP

$$-y''(x) - \frac{1}{x}y'(x) = \frac{(e^x)}{100} - \frac{y^3}{30} + \frac{xy'}{5}, \tag{34}$$

$$y'(0) = 0, \quad y(1) = 0.6y\left(\frac{2}{5}\right). \tag{35}$$

Here solution of nonlinear three point SBVP (32)–(33) has $\alpha_0 = -1$ and $\beta_0 = 1$ as lower and upper bounds, respectively. This is a well ordered case. The nonlinear sources term satisfies the conditions $(F_1)$, $(F_2)$ and $(F_3)$ in domain $\widetilde{D}$. Here Lipschitz constant are $M_2 = \frac{1}{10}$ and $N = \frac{1}{5}$. From Fig. 2, it is clear that we can find out a range of $\lambda < 0$ such that

**Fig. 2** Plot of $(H_0')$




$$\lambda \leq \min \left\{ -M_2, -M_2 - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M_2}, -\frac{M_2}{1-N}, \inf \left( \frac{f(x, \beta, x\beta') - f(x, \alpha, x\alpha')}{\beta - \alpha} \right) \right\},$$

i.e., $\lambda \leq -0.186332$. So that $(H_0')$, is satisfied. Thus Theorem 4.2 is applicable here.

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